On Ben Johnston's Notation and the Performance Practice of Extended Just Intonation

by Marc Sabat

1. Introduction: Two Different E's

Like the metric system, the modern tempered tuning which divides an octave into 12 equal but irrational proportions was a product of a time obsessed with industrial standardization and mass production. In Schönberg's words: *a reduction of natural relations to manageable ones.* Its ubiquity in Western musical thinking, epitomized by the pianos which were once present in every home, and transferred by default to fixed-pitch percussion, modern organs and synthesizers, belies its own history as well as everyday musical experience.

As a young musician, I studied composition, piano and violin. Early on, I began to learn about musical intervals, the sound of two tones *in relation to each other.* Without any technical intervention other than a pitch-pipe, I learned to tune my open strings to the notes G - D - A - E by playing two notes at once, listening carefully to eliminate beating between overtone-unisons and seeking a stable, resonant sound-pattern called a "perfect fifth".

At the time, I did not know or need to know that this consonance was the result of a simple mathematical relationship, that the lower string was vibrating twice for every three vibrations of the upper one. However, when I began to learn about placing my fingers on the strings to tune other pitches, the difficulties began. To find the lower E which lies one whole step above the D string, I needed to place my first finger down. Since the open E is two strings away, I could not rely on hearing an octave to check if my pitch was true. Instead, I had two other possibilities: to tune the major sixth from G to E or the perfect fourth from E to A.

Anyone who tries this and listens for the most consonant tuning of overtones will easily find two *different* E's. But the piano only has one, traditional notation only has one, and when musicians speak about tuning they usually say that something is simply "in tune" or "out of tune".

So which E is "right"? In spite of the clear evidence of my own senses, the force of mass belief in a cultural system encouraged me to somehow try and decide in favor of *one* E, and for many years I kept trying to do so, thinking that there was something wrong with my oversensitivity. It wasn't until my twenties, when I accidentally came across Harry Partch's book "Genesis of a Music" and Hermann von Helmholtz's "On the Sensations of Tone as a Physiological Basis for the Theory of Music", that I began to realize that in fact my perception *was* accurate, and the piano and the conventional music notation insufficient.

2. Musical Intervals and Frequency Ratios

The reason for such microtonal variations is based on very simple mathematics. Pitches are periodic vibrations, and natural intervals are patterns produced by a *ratio of frequencies.* The simpler the pattern, the more consonant the resulting combination. Thus, in tuning my open strings, I had found the ratio 3/2. In tuning the perfect fourth A - E, I was hearing the ratio 4/3; for the major sixth G - E, I had two choices: taking the same E against G produces the ratio 27/16, which by virtue of its larger numbers sounds more complex and dissonant; by lowering E slightly one finds the ratio 5/3, which sounds more simple and consonant. However, the ratio from this lower E to the open A string then becomes 27/20, which has a complex and dissonant sound.[1]

Using ratios one can *measure the difference* between these two E's, simply by *dividing the two ratios*:

(27/16) ÷ (5/3) = (81/80) = one *Syntonic Comma*

In this example, not only are two slightly different pitches being produced, but even more importantly each E brings both a *consonance* and a *dissonance* into play. It is exactly this consequence, the necessity of accepting *well-tuned dissonances,* which for a long time presented an obstacle to making Just Intonation (JI for short) a conscious part of musical practice. Instead, in Western music intonation is kept mysterious, something which good musicians are expected to achieve intuitively.

3. A Brief History of Intonation in European Music

In the 9th through the 12th centuries across Europe there developed a vocal music based on diatonic modes harmonized in fifths, fourths, and octaves. The most natural intonation of such modes, derived from ancient Greek harmonic theory via Boethius, was achieved by constructing a series of fifths from one note to the next, a so-called *Pythagorean tuning.* As vocal polyphony developed, the harmonies between voices began to include major and minor thirds and sixths, and the classical Pythagorean tuning was expanded to a *Ptolemaic*[2] system, including the ratios 5/4, 6/5, 5/3, 8/5. In practice, this meant that singers and instrumentalists would make small comma-adjustments to facilitate the weaving of consonances and dissonances in the music.

[1] To add musical intervals, one must *multiply* the respective ratios. So, for example, the two perfect fifths G - D and D - A added together produce a ninth: (3/2) • (3/2) = (9/4).
To subtract musical intervals, one must *divide* the respective ratios. Thus the ninth less a perfect fourth produces the higher of two E's: (9/4) ÷ (4/3) = (27/16). The ninth less a consonant major sixth produces a dissonant large fourth: (9/4) ÷ (5/3) = (27/20).

[2] Claudius Ptolemy was one the earliest theorists to outline the sequence of intervals which defines a major scale in triadic (5-Limit) JI. This set of pitches may be constructed from any tonic by combining the subdominant, tonic, and dominant triads, each tuned in the proportion 4:5:6.

In the 16th century, the theorist Zarlino codified this ongoing praxis, stating that the natural intonation practiced by singers and instruments of flexible pitch was based on consonant intervals that could be expressed as ratios of the first six whole numbers (a set he called the *senario).* At the same time, Zarlino was the first to precisely describe the practice of temperament on fretted or keyed instruments of fixed pitch. His *2/7-comma Meantone*[3] *Temperament* proposes a geometrically constructed compromise which mistunes all of the common consonances into gently beating approximations. This system allowed a keyboard instrument to come close to natural intervals without having a mechanically unmanageable plethora of keys. Zarlino even described his meantone temperament in theological terms, arguing that such an elegantly constructed compromise from the ideal intervals mirrored the imperfection of human condition in contrast to the divine.

Equal Temperament was proposed around the same time as suitable for fretted instruments, keeping in mind that experienced gamba players were always able to make subtle adjustments by bending the strings. Perhaps most significant in this time and the two centuries following was an acceptance of *many models of tuning simultaneously:* pure intervals and tempered intervals, requiring *multiple interpretations* of the musical symbols flat, natural, sharp.

In Pythagorean tuning, flats are *lower* than enharmonically spelled sharps; in Equal Temperament flats are *equal* to the sharps; in Meantone Temperament flats are *higher* than sharps. In JI, *each note has various possible tunings based on its harmonic context.* Indian music theory explicitly describes these variations of tuning as *srutis,* which are understood as microtonal variations of the 12 chromatic pitch-classes. Such microtonal shadings and spectral sound-aggregates are increasingly prevalent in contemporary written music, and this has inspired various attempts to find appropriate notations for them.

4. Helmholtz and Ellis

In the 19th century, Helmholtz argued for JI based on correctly tuned fifths and thirds. In his book he uses the conventional note-names to represent a series of perfect fifths. So, for example, the common open strings are written conventionally as C · G · D · A · E. To make a simple consonant interval with C and G (thereby completing a triad tuned in the proportion 4:5:6) E *must be lowered* by the ratio 81/80 (one Syntonic comma), as described above. If this lowering is explicitly notated, it becomes possible to write up correctly tuned just major thirds *above any note in the series of fifths.* By a similar logic, the just major thirds *below any note* may be written, simply by inverting the symbol used.[4] Such a system can most elegantly be understood by drawing a two-dimensional tone lattice, following a model which Leonhard Euler first

[3] Meantone refers to the tempering of the two whole-tones 9/8 and 10/9, which together make a pure major third 5/4. Instead, this third is closely approximated by choosing two equal-sized *mean* tones.

proposed in his writings about mathematics and music theory. On one axis notes are related by perfect fifths and fourths, and on a second perpendicular axis notes are related by major thirds or minor sixths.

Helmholtz' English translator Alexander Ellis recognized that once various tunings of pitches are introduced it becomes important to have a fine ruler for measuring these differences, simply to be sure which is a little higher and which is a little lower. As musicians, we are already used to this logic, derived from the Greek theorist Aristoxenus: we speak of wholetones, semitones, quarter-tones, even sixth-tones. Ellis proposed a formula which would create a metric division of each tempered semitone into 100 units, called *cents.* In effect, this is a division of the octave (2/1) into 1200 equal-sounding parts.

4. Calculating Cents

To convert any ratio to a value in cents, the following formula may be applied:

cents = 1200 • log (**ratio**) / log (2)

For example, to calculate the size of the just perfect fifth:

1.) Write the ratio as a fraction 3/2, with the larger number divided by the smaller (this simply considers the ratio as *an interval measured upward).*

2.) Calculate the logarithm of 3/2 divided by the logarithm of 2, and multiply by 1200.

3.) The rounded result should be about 702 cents, which is equivalent to 7.02 tempered semitones.

This means that a 3/2 frequency ratio is larger than 7 tempered semitones by a tiny amount, 2/100 of a tempered semitone (a *just noticeable difference).* Using this formula, it is possible to find out that one Syntonic Comma (81/80) is equal to around 21.5 cents (somewhat less than an eighth-tone).

[4] The Extended Helmholtz-Ellis JI Pitch Notation, conceived by Marc Sabat and Wolfgang von Schweinitz, is a system of accidental signs based on Helmholtz' method. It allows all frequency ratios to be written within the common five-line staff notation. To indicate alteration by a Syntonic Comma, an arrow pointing upward or downward is attached to the conventional flat, natural, or sharp sign. For more information, please visit **www.plainsound.org**

5. Learning the 5-Limit Intervals

In the early 20th century, American composer and theorist Harry Partch became fascinated with the sound of just intervals tuned by ear, and not only the classical consonances constructed from fifths and thirds. In his music he began to also incorporate intervals derived from the 7th and 11th partials of the harmonic series. To organize this expanded pitch system Partch introduced the concept of various *prime limits:* he referred to Pythagorean tuning as 3-Limit, Ptolemaic tuning as 5-Limit, and his own system as 11-Limit. This idea is based on the fact that all ratios can be broken down into products of prime numbers. For each prime number there are characteristic intervals – fundamental building blocks – which must be learned by ear to tune within that prime limit.

To accurately realize Pythagorean tuning, a musician needs to learn intervals based on the primes 2 and 3: the perfect fifth 3/2, the perfect fourth 4/3, the major ninth 9/4 and the wholetone 9/8. By combining these intervals, all other 3-Limit pitches may be found.

A 5-Limit system is based on the primes 2, 3 and 5. In addition to the Pythagorean intervals, there are now new consonances (intervals tuneable by ear) based on the prime number 5. For example, the major third 5/4, the minor third 6/5, the major sixth 5/3, the minor sixth 8/5, the minor seventh 9/5 and the major seventh 15/8. As well, 5-Limit tuning introduces a slightly smaller wholetone with the ratio 10/9. To become familiar with these intervals, it is useful to calculate their sizes in cents, to produce each of them with the help of an electronic tuner, and to tune them to an overtone-rich sustained drone (i.e. an Indian sruti box, a reed organ or an accordion).

interval name	**limit**	**ratio**	**cents**	**tempered**	**deviation**
minor wholetone	5	10/9	182	200	–18
major wholetone	3	9/8	204	200	+4
minor third	5	6/5	316	300	+16
major third	5	5/4	386	400	–14
perfect fourth	3	4/3	498	500	–2
perfect fifth	3	3/2	702	700	+2
minor sixth	5	8/5	814	800	+14
major sixth	5	5/3	884	900	–16[i]
minor seventh	5	9/5	1018	1000	+18
major seventh	5	15/8	1088	1100	–12
octave	2	2/1	1200	1200	0
major ninth	3	9/4	1404	1400	+4

With the exception of the two whole-tones, each of these intervals can be tuned very accurately simply by listening to its sound, by trying to minimize beating between common partials, and focussing the spectral clarity of combination tones.[5] Each has a distinctive sound which is readily learned and may then be used in combinations to construct more complex 5-Limit sounds.

6. Ben Johnston's Notation

Following in Partch's tradition, the American composer Ben Johnston has developed a set of microtonal accidentals which he uses to represent JI intervals. In his system, the seven white notes C D E F G A B are tuned as *consonant major triads* above F, C and G (subdominant, tonic, dominant in the key of C):

D	
G	B
C	E
F	A

In the diagram, left-to-right represents a 4:5 major third, and upwards represents a 2:3 perfect fifth. This means that the interval D - A is not a perfect fifth, rather it is one Syntonic Comma smaller, that is:

D - A = (3/2) ÷ (81/80) = (40/27)

To notate the Syntonic Comma, Johnston employs the signs + and – so that the perfect fifth above D is written as A+. Extending the matrix:

D+			
G+	B+		
C+	E+		
F+	A+		
	D		
	G	B	
	C	E	
	F	A	
		D–	
		G–	B–
		C–	E–
		F–	A–

[5] Here the difference tone method is particularly useful, and may be calculated simply by subtracting the numerator and denominator of the ratio and working out its pitch in a harmonic series. Thus, in the major sixth 5/3, the difference tone is 5–3=2. If 5 is a B and 3 is a D, then 2 is a G (a fifth below D).

Each block of seven pitches is a 5-Limit major scale, and each block is one Syntonic Comma higher than the previous one.

Continuing with this logic, Johnston defines flats in such a way that the three minor triads on F, C and G are tuned in 5-Limit JI. In a major triad, the major third 4:5 is on the bottom, and the minor third 5:6 is on top, creating a proportion 4:5:6. The minor triad inverts this relationship, placing the minor third 5:6 = 10:12 on the bottom, and the major third 4:5 = 12:15 on top, creating a proportion 10:12:15. Thus, the difference between E and Eb is the difference between a just major third and a just minor third:

$(5/4) \div (6/5) = (25/24) = 71$ cents

This interval is called the 5-Limit *chromatic semitone*. A flat lowers by this amount, a sharp raises by the same ratio. Based on this definition, it is now possible to expand the matrix by including sharps and flats:

	D+				
	G+	B+	D#+		
	C+	E+	G#+	B#+	
Db	F+	A+	C#+	E#+	
Gb	Bb	D	F#+	A#+	
Cb	Eb	G	B	D#	
Fb	Ab	C	E	G#	B#
	Db–	F	A	C#	E#
	Gb–	Bb–	D–	F#	A#
	Cb–	Eb–	G–	B–	
	Fb–	Ab–	C–	E–	
			F–	A–	

5-Limit intervals may be thought of as 2-dimensional “chess moves” within this lattice, as indicated in the following table.

interval name	**ratio**	**steps left (–) or right (+) [5^]**	**steps down (–) or up (+) [3^]**
chromatic semitone	25/24	+2	–1
diatonic semitone	16/15	–1	–1
minor wholetone	10/9	+1	–2
major wholetone	9/8	0	+2
minor third	6/5	–1	+1
major third	5/4	+1	0
perfect fourth	4/3	0	–1
augmented fourth	45/32	+1	+2
diminished fifth	64/45	–1[ii]	–2[iii]
perfect fifth	3/2	0	+1
minor sixth	8/5	–1	0
major sixth	5/3	+1	–1
minor seventh	9/5	–1	+2
major seventh	15/8	+1[iv]	+1
octave	2/1	0	0
major ninth	9/4	0	+2

Note the addition of four new intervals: the *chromatic semitone* 25/24 (difference between a major and minor third), the *diatonic semitone* 16/15 (difference between a perfect fourth and a major third), the *augmented fourth* 45/32 (a major third combined with a major whole-tone 9/8) and its inversion the *diminished fifth* 64/45.

6. Extended Just Intonation

For higher prime limits, Johnston has coined the term *Extended Just Intonation.* To become familiar with these more distant, unfamiliar, and in some cases dissonant intervals, the most effective approach is to identify characteristic easily tuned relationships for each prime. These can serve as building blocks to reach the more harmonically distant pitches following a similar lattice-based approach to the one outlined above.

For the 7-Limit, perhaps the easiest interval to accurately produce is the narrow minor tenth 7/3 (1467 cents, or a deviation of –33 from the tempered minor tenth). The septimal minor seventh 7/4 and the septimal diminished fifth 7/5 are also relatively easily learned. In Johnston's notation, septimal intervals *above* a note are indicated by adding a small 7 accidental. An inverted 7 simply means that the septimal interval was generated *downward*.

The 11-Limit is reached most easily by means of the interval 11/4, which is almost exactly one octave plus a perfect fourth plus a quarter-tone (1751 cents). 11/6 (the neutral seventh) is also readily learned. Johnston indicates these pitches with an arrow upward (interval above) or downward (interval below).

The 13-Limit is perhaps the most difficult to accurately learn. It is characterized by the neutral sixth 13/8 (an interval which may be found approximately 1/3 of the way between a minor sixth and a major sixth). This relation is more easily heard in wide position (13/4). In higher registers, it may be possible to hear the difference tone produced by a 13/8, particularly in the progression 5/3 · 13/8 · 8/5, in which case the difference tones descend by semitones. Several other easier 13-Limit sounds are the neutral ninth 13/6, the small major seventh 13/7, and the major third-plus-quarter-tone 13/10. Johnston writes these intervals with a small added 13 or inverted 13.[6]

In general, realizing pitches in extended JI requires that a performer always think in terms of intervallic relationships to other pitches:

1.) What other pitches are sounding or have recently sounded?
2.) How are they most simply related to each other?
3.) Which prime building blocks can be used to construct a well-tuned relationship?

It is exactly this kind of extended harmonic awareness that allows such complex pitch-relationships to become comprehensible in the imagination of the performer, and thus to be transmitted with maximum clarity to the ears of the listener.

[6] To avoid confusion in the case of an inverted 13, it is useful to bear in mind its resemblance to 31!

6. Notes about transpositions and cents calculation in Johnston's music

In the case of transposing instruments in Johnston's music, it is important to determine which JI interval is being used for the notated transposition. For example, in "O Waly Waly Variations", the two saxophones in Bb are written in G Major, sounding F Major, a 9/8 major wholetone lower (plus the usual octave transpositions as needed). Thus, the first two notes are written as A+ and D but sound as G and C. The two Eb instruments are written in D– Major, sounding F Major, a 6/5 minor third higher (plus the usual octave transpositions as needed). Thus, the first two notes are written as A and D–, sounding C and F. The resulting harmonies are 3-Limit perfect fifths and fourths.

If performers wish to calculate cents deviations, the first step is to decide which pitch will be 0 cents (reference). Then, it is necessary to calculate ratios from this reference point (for very exact cents).

A shorthand method is to simply calculate the cents deviations of each of the seven white notes, and then add or subtract the cents deviations of each additional accidental.

For example, if sounding F is chosen as a reference, here are the ratios and cents deviations of the seven white pitches:

note	**ratio to F**	**cents**	**deviation**
F	1/1	0	0
G	9/8	204	+4
A	5/4	386	–14
B	45/32	590	–10
C	3/2	702	+2
D	27/16	906	+6
E	15/8	1088	–12

The deviations must be appropriately offset should another pitch be chosen as 0 cents.

Thus, if A is 0 cents, as is usual on electronic tuners, then all of the deviations in this table must be corrected by +14 cents. F becomes +14, G +18, and so on. Thus, the open strings of the orchestral string instruments will be written with the notes C– G– D– A E, and tuned to –6, –4, –2, 0, and +2 cents respectively, as Johnston does in his string quartets for the most part.

Following is a table of accidental alteration amounts:

sign / prime limit	**ratio**	**cents**
+ / –	81/80	22
b / #	25/24	71
7	36/35	49
11 (arrow)	33/32	53
13	65/64	27

It may also be helpful to recall which interval is most commonly altered by each of these signs. The + and – signs provide comma corrections, often occurring between D and A (to make a consonant fifth, D A+ or D– A); similarly between Bb and F to make a consonant fifth; and between D and F or F# to make a consonant third D F+ or D F#+. The b and # signs are conventional but substantially *smaller* than their equal tempered equivalents.

The 7 sign alters a 9/5 interval (the 5-Limit minor seventh) *downward* by approximately a quarter-tone to produce the septimal minor seventh 7/4.

The 11 sign (arrow-up) alters a perfect fourth *upward* by approximately a quarter-tone to produce the 11/8 ratio (which is most easily learned in wide position as the ratio 11/4 mentioned above).

The 13 sign alters the 5-Limit minor sixth *upward* by approximately a sixth-tone to produce the neutral sixth 13/8.

As long as these relationships are kept in mind, the exact harmonic relationships between successive and simultaneous pitches can readily be determined.

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